

A generalization of Marstrand's theorem and a geometric application

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Abstract

In this paper we prove two general results related to Marstrand's projection theorem in a quite general formulation over separable metric spaces under a suitable transversality hypothesis (the “projections” are in principle only measurable) - the result is flexible enough to, in particular, recover most of the classical Marstrand-like theorems. Our proofs use simpler tools than many classical works in the subject, where some techniques from harmonic analysis or special geometrical structures on the spaces are used. Also we show a geometric application of our results.

1 Introduction

Let X, Y be separable metric spaces, (Λ, \mathcal{P}) a probability space and $\pi : \Lambda \times X \rightarrow Y$ a measurable function. Informally, one can think of $\pi_\lambda(\cdot) = \pi(\lambda, \cdot)$ as a family of projections parameterized by λ . We assume that for some positive real numbers α, κ and C the following transversality property is satisfied:

$$\mathcal{P}[\lambda \in \Lambda : d(\pi_\lambda(x_1), \pi_\lambda(x_2)) \leq \delta d(x_1, x_2)^\alpha] \leq C\delta^\kappa \quad (1.1)$$

for all $\delta > 0$ and all $x_1, x_2 \in X$. In most examples $\kappa = \dim Y$.

We call the measure η on Y κ -regular if it is Borel, upper regular by open sets and

$$\liminf_{r \rightarrow 0} \eta(B(y, r))/r^\kappa > 0 \text{ for all } y \in Y.$$

Lebesgue measure in Euclidean spaces are strongly regular, in the sense that the limit is uniformly positive. In those cases we say that η is *strongly κ -regular*.

We are able to prove general versions of Marstrand theorem in this context. Assuming X to be an analytic subset of a complete separable metric space, we have:

Theorem 1.1. $\dim \pi_\lambda(X) \geq \min(\kappa, \dim X/\alpha)$ for a.e. $\lambda \in \Lambda$.

In particular, if $\kappa = \dim Y$ and assuming in the case when $\dim X < \alpha\kappa$ that the maps π_λ are α -Hölder, then $\dim \pi_\lambda(X) = \min(\kappa, \dim X/\alpha)$ for a.e. $\lambda \in \Lambda$.

*The first author was supported by the Balzan Research Project of J.Palis and by INCTMat and PCI projects. The second author was supported by the Balzan Research Project of J.Palis and by CNPq.

Theorem 1.2. *Suppose $\dim X > \alpha\kappa$.*

(i) *If η is κ -regular, then $\eta(\pi_\lambda(X)) > 0$ for a.e. $\lambda \in \Lambda$.*

(ii) *If η is strongly κ -regular and σ -finite, then $\int_\Lambda \eta(\pi_\lambda(X))^{-1} d\mathcal{P} < +\infty$.*

Remark 1.3. Those Theorems recover the Solomyak formulation in [Sol98, Theorem 5.1] Solomyak's scheme has the interesting feature of considering the exponent α depending continuously on the parameter λ . However, Solomyak makes the hypothesis that the maps π_λ are α -Hölder, which we don't need for our results (in particular for Theorem 1.2). This gives us more flexibility for applications. As Solomyak, we also use the ideas of Mattila in [Mat95] to prove this variation of Marstrand's Theorem. However, some of them can not be extended to our more general context. In those cases our modifications stay elementary enough.

2 Preliminaries

For X separable metric space, let

$$\mathcal{M}^s(X) = \{\mu : \mu \text{ Borel measure on } X \text{ with } 0 < \mu(X) < \infty \text{ such that there is } c > 0 \text{ satisfying } \mu(B(x, r)) \leq cr^s \text{ for all } x \in X, r > 0\}.$$

we have $\mathcal{M}^s(X) \neq \emptyset \Rightarrow \mathcal{H}^s(X) > 0$. Indeed, if $\mu \in \mathcal{M}^s(X)$ and E_1, E_2, \dots is any covering of X , picking $x_j \in E_j, j = 1, 2, \dots$ we have $B(x_j, d(E_j)) \supset E_j$ and therefore

$$\sum_j d(E_j)^s \geq \sum_j \frac{1}{c} \mu(B(x_j, d(E_j))) \geq \sum_j \frac{1}{c} \mu(E_j),$$

hence $\mathcal{H}^s(X) \geq \mu(X)/c > 0$. The converse is also true for analytic sets in complete separable metric space, see for instance [Mat95, Theorem 8.17]. We give an elementary proof for the same result starting from Howroyd [How95].

Lemma 2.1. *If X is an analytic subset of a complete separable metric space, then $\mathcal{H}^s(X) > 0 \Rightarrow \mathcal{M}^s(X) \neq \emptyset$.*

Proof. By Corollary 7 in [How95], every analytic subset of a complete separable metric space which has positive (or infinite) Hausdorff measure of dimension s contains a compact set which has finite and positive Hausdorff measure of dimension s , and so we may assume X compact with $0 < \mathcal{H}^s(X) < \infty$. Let $M > 10^s$ and $\delta > 0$ such that $\mathcal{H}_\delta^s(X) > 10^s \mathcal{H}^s(X)/M$ and consider the family

$$\mathcal{B} := \{B(x, r) : 10r \leq \delta, \mathcal{H}^s(B(x, r)) \geq Mr^s\}.$$

By the Vitaly covering theorem we get disjoint balls $B(x_j, r_j) \in \mathcal{B}, j = 1, 2, \dots$ such that $\cup_{B \in \mathcal{B}} B \subset \cup_j B(x_j, 5r_j) =: A$, hence

$$\mathcal{H}_\delta^s(A) \leq 10^s \sum_j r_j^s \leq \frac{10^s}{M} \sum_j \mathcal{H}^s(B(x_j, r_j)) \leq 10^s \mathcal{H}^s(X)/M$$

and therefore $\mathcal{H}^s(X \setminus A) \geq \mathcal{H}_\delta^s(X \setminus A) \geq \mathcal{H}_\delta^s(X) - \mathcal{H}_\delta^s(A) > 0$. Obviously, \mathcal{H}^s restricted to $X \setminus A$ is in $\mathcal{M}^s(X)$ for $c = \max(M, 10^s \mathcal{H}^s(X)/\delta^s)$. \square

Let μ be a finite Borel measure on X . The s -energy of μ is

$$I_s(\mu) = \int \int \frac{d\mu(x_1)d\mu(x_2)}{d(x_1, x_2)^s}.$$

Finiteness of energy is closely related to measures in $\mathcal{M}^s(X)$. For instance, if $I_s(\mu) < \infty$, then $\sup_{r>0} |\mu(B(x, r))|/r^s \leq \int d(x, x_2)^{-s} d\mu(x_2) < \infty$ for μ -a.e. $x \in X$. If M is large enough such that the Borel set $B = \{x : \sup_{r>0} |\mu(B(x, r))|/r^s \leq M\}$ has positive μ -measure, then $\nu = \mu|_B$ is in $\mathcal{M}^s(X)$. In fact, if $B(x, r) \cap B = \emptyset$ then $\nu(B(x, r)) = 0$, otherwise, if $z \in B(x, r) \cap B$, then $B(x, r) \subset B(z, 2r)$ and therefore

$$\nu(B(x, r)) \leq \nu(B(z, 2r)) \leq 2^s M r^s.$$

On the other hand if $\mu \in \mathcal{M}^s(X)$, then $I_t(\mu) < \infty$ for all $0 \leq t < s$. In fact,

$$\begin{aligned} \int d(x_1, x_2)^{-t} d\mu(x_2) &= \int_0^\infty \mu[x_2 : d(x_1, x_2)^{-t} \geq u] du \\ &= \int_0^\infty \mu(B(x_1, u^{-\frac{1}{t}})) du \leq \mu(X) + c \int_1^\infty u^{-\frac{s}{t}} du < \infty. \end{aligned}$$

Then, if X is an analytic subset of a complete separable metric space, we have:

$$\begin{aligned} \dim X &:= \sup \{s : \mathcal{H}^s(X) > 0\} \\ &= \sup \{s : \exists \mu \text{ Borel measure with } 0 < \mu(X) < \infty \text{ and } I_s(\mu) < \infty\}. \end{aligned} \quad (2.1)$$

2.1 A scheme to compare measures.

We present here an elementary alternative to the methods of differentiation theory of Radon measures in Euclidean spaces that actually can not be extended to the general setting we are interested.

Given any Borel measure ν on Y we define its κ -density as

$$D^\kappa \nu(y) := \liminf_{r \rightarrow 0} \nu(B(y, r))/r^\kappa.$$

Lemma 2.2. *Let ν_1 and ν_2 be Borel measures, ν_2 upper regular and $0 < s, t < \infty$. If $D^\kappa \nu_1(y) \leq s$ and $D^\kappa \nu_2(y) \geq t$ for all $y \in A$, then $\nu_1(A) \leq 5^\kappa s t^{-1} \nu_2(A)$.*

Proof. For any $\varepsilon > 0$ and any $U \supset A$ open set, by the Vitali covering theorem we get disjoint balls $B(x_j, r_j) \subset U$ with $\nu_1(B(x_j, 5r_j)) \leq (s+\varepsilon)(5r_j)^\kappa$ and $\nu_2(B(x_j, r_j)) \geq (t-\varepsilon)r_j^\kappa$ for $j = 1, 2, \dots$ such that $A \subset \cup_j B(x_j, 5r_j)$, then

$$\nu_1(A) \leq \sum_j \nu_1(B(x_j, 5r_j)) \leq 5^\kappa \frac{s+\varepsilon}{t-\varepsilon} \sum_j \nu_2(B(x_j, r_j)) \leq 5^\kappa \frac{s+\varepsilon}{t-\varepsilon} \nu_2(U).$$

The inequality now follows from the upper regularity of ν_2 and making $\varepsilon \rightarrow 0$. \square

This lemma has a strong version for Radon measures in Euclidean spaces, see [Mat95, Lemma 2.13], where for instance the 5^κ -factor does not appear.

Proposition 2.3. *Let ν and η be Borel measures such that η is upper regular and $D^\kappa \nu(y) < \infty$ for ν -a.e. $y \in Y$.*

(i) *If $D^\kappa \eta(y) > 0$ for all $y \in Y$, then $\nu \ll \eta$.*

(ii) *If $D^\kappa \eta(y) \geq c > 0$ for all $y \in Y$ and η is σ -finite, then $d\nu/d\eta \leq 5^\kappa c^{-1} D^\kappa \nu$ for η -a.e..*

Proof. (i) Let $A \subset Y$ with $\eta(A) = 0$. Let $P_M = \{y : D^\kappa \nu(y) < M, D^\kappa \eta(y) > M^{-1}\}$. By Lemma 2.2 $\nu(A \cap P_M) \leq 5^\kappa M^2 \eta(A \cap P_M) = 0$. Making $M \rightarrow \infty$ we get $\nu(A) = 0$.

(ii) Let $B \subset Y$ any Borel set. For $1 < t < \infty$ let $B_p = \{y \in B : t^p \leq D^\kappa \nu(y) < t^{p+1}\}$, $p \in \mathbb{Z}$. By the Lemma 2.2 $\nu(B_p) \leq 5^\kappa t^{p+1} c^{-1} \eta(B_p)$ and $\nu(\{D^\kappa \nu = 0\}) = 0$, so we have

$$\begin{aligned} \nu(B) &= \sum_{p \in \mathbb{Z}} \nu(B_p) \leq \sum_{p \in \mathbb{Z}} 5^\kappa t^{p+1} c^{-1} \eta(B_p) \\ &\leq t 5^\kappa c^{-1} \sum_p \int_{B_p} D^\kappa \nu d\eta \leq t 5^\kappa c^{-1} \int_B D^\kappa \nu d\eta. \end{aligned}$$

Making $t \rightarrow 1$, we get $\nu(B) \leq 5^\kappa c^{-1} \int_B D^\kappa \nu d\eta$ for all Borel set $B \subset Y$. \square

This Proposition is in some sense related to Theorem 2.12 in [Mat95].

3 Proof of the Theorems

Let X, Y be separable metric spaces, (Λ, \mathcal{P}) a probability space and $\pi : \Lambda \times X \rightarrow Y$ a measurable function satisfying the transversality property in 1.1.

Given any finite Borel measure μ on X let $\nu_\lambda = (\pi_\lambda)_* \mu$. Note that ν_λ are also Borel measures with $\nu_\lambda(\pi_\lambda(X)) = \mu(X)$.

The following two lemmas come naturally from ideas of Theorem 9.3 and Theorem 9.7 in [Mat95].

Lemma 3.1.

$$\int_\Lambda I_t(\nu_\lambda) d\mathcal{P} \leq C_{t,\kappa} I_{\alpha t}(\mu) \text{ for all } 0 \leq t < \kappa, \text{ where } C_{t,\kappa} = 1 + \frac{Ct}{\kappa - t} < +\infty.$$

Proof. Notice that $I_t(\nu_\lambda) = \int \int \frac{d\mu(x_1) d\mu(x_2)}{d(\pi_\lambda(x_1), \pi_\lambda(x_2))^t}$ then, by Fubini's theorem, we have

$$\int_\Lambda I_t(\nu_\lambda) d\mathcal{P} = \int \int \left[\int_\Lambda \left(\frac{d(\pi_\lambda(x_1), \pi_\lambda(x_2))}{d(x_1, x_2)^\alpha} \right)^{-t} d\mathcal{P} \right] \frac{d\mu(x_1) d\mu(x_2)}{d(x_1, x_2)^{\alpha t}},$$

and

$$\begin{aligned} \int_\Lambda \left(\frac{d(\pi_\lambda(x_1), \pi_\lambda(x_2))}{d(x_1, x_2)^\alpha} \right)^{-t} d\mathcal{P} &= \int_0^\infty \mathcal{P} \left[\lambda \in \Lambda : \left(\frac{d(\pi_\lambda(x_1), \pi_\lambda(x_2))}{d(x_1, x_2)^\alpha} \right)^{-t} \geq u \right] du \\ &= \int_0^\infty \mathcal{P} \left[\lambda \in \Lambda : \frac{d(\pi_\lambda(x_1), \pi_\lambda(x_2))}{d(x_1, x_2)^\alpha} \leq u^{-\frac{1}{t}} \right] du \\ &\leq 1 + C \int_1^\infty u^{-\frac{\kappa}{t}} du = C_{t,\kappa} < \infty. \end{aligned}$$

\square

Proof of Theorem 1.1. It follows from equation (2.1) and Lemma 3.1. \square

Lemma 3.2.

$$\int_{\Lambda} \int D^{\kappa} \nu_{\lambda} d\nu_{\lambda} d\mathcal{P} \leq CI_{\alpha\kappa}(\mu).$$

Proof. Using Fatou's lemma

$$\int \liminf_{r \rightarrow 0} \frac{\nu_{\lambda}(B(y, r))}{r^{\kappa}} d\nu_{\lambda}(y) \leq \liminf_{r \rightarrow 0} \frac{1}{r^{\kappa}} \int \nu_{\lambda}(B(y, r)) d\nu_{\lambda}(y), \text{ and}$$

$$\int \nu_{\lambda}(B(y, r)) d\nu_{\lambda}(y) = \nu_{\lambda} \times \nu_{\lambda}[(y_1, y_2) : d(y_1, y_2) \leq r] = \mu \times \mu[(x_1, x_2) : d(\pi_{\lambda}(x_1), \pi_{\lambda}(x_2)) \leq r],$$

then, by Fubini's theorem and (1.1)

$$\begin{aligned} \int_{\Lambda} \int D^{\kappa} \nu_{\lambda} d\nu_{\lambda} d\mathcal{P} &\leq \liminf_{r \rightarrow 0} \iint \frac{1}{r^{\kappa}} \mathcal{P}[\lambda \in \Lambda : d(\pi_{\lambda}(x_1), \pi_{\lambda}(x_2)) \leq r] d\mu(x_1) d\mu(x_2) \\ &\leq CI_{\alpha\kappa}(\mu). \end{aligned}$$

\square

Proof of Theorem 1.2. (i) follows from Lemma 3.2, Lemma 2.1 and Proposition 2.3(i), since if $\nu_{\lambda} \ll \eta$ with $\nu_{\lambda}(\pi_{\lambda}(X)) = \mu(X) > 0$ then $\eta(\pi_{\lambda}(X)) > 0$.

Analogously with Proposition 2.3(ii) we have $d\nu_{\lambda}/d\eta \in L^2(\eta)$ for a.e. $\lambda \in \Lambda$ with

$$\int_{\Lambda} \left\| \frac{d\nu_{\lambda}}{d\eta} \right\|_{L^2(\eta)}^2 d\mathcal{P} \leq 5^{\kappa} c^{-1} CI_{\alpha\kappa}(\mu) < \infty.$$

Part (ii) now follows from the Cauchy-Schwarz's inequality

$$\mu(X)^2 = (\nu_{\lambda}(\pi_{\lambda}(X)))^2 = \left(\int_{\pi_{\lambda}(X)} \frac{d\nu_{\lambda}}{d\eta} d\eta \right)^2 \leq \eta(\pi_{\lambda}(X)) \left\| \frac{d\nu_{\lambda}}{d\eta} \right\|_{L^2(\eta)}^2,$$

which implies

$$\int_{\Lambda} \eta(\pi_{\lambda}(X))^{-1} d\mathcal{P} \leq \mu(X)^{-2} \int_{\Lambda} \left\| \frac{d\nu_{\lambda}}{d\eta} \right\|_{L^2(\eta)}^2 d\mathcal{P} \leq 5^{\kappa} c^{-1} \mu(X)^{-2} CI_{\alpha\kappa}(\mu) < +\infty.$$

\square

4 An Example: A version of Marstrand's theorem in non-positive curvature

4.1 Setting

Let M be a Riemannian manifold with metric $\langle \cdot, \cdot \rangle$. A line in M is a geodesic defined for all parameter values and minimizing distance between any of its points, that is, $\gamma : \mathbb{R} \rightarrow M$ is an isometry. If M is a manifold of dimension n , simply connected and of non-positive

curvature, then the space of lines leaving a point p can be seen as a sphere of dimension $n - 1$. So, in the case of surfaces the set of lines leaving a point p can be identified with S^1 in the space tangent $T_p M$ of the point p . Therefore, in each point on the surface the set of lines can be oriented and parametrized by $[0, \pi)$ and endowed with the Lebesgue measure. Thus, using the previous identification, we can talk about almost every line through a point of M (cf. [BH99]). In the conditions above, Hadamard's theorem states that M is diffeomorphic to \mathbb{R}^n , (cf. [dC08]).

Given $x \neq y$, denote by $[x, y]$ the segment of geodesic joining x and y and for three different points x, y, z , denote by $\angle_y(x, z)$ the angle in y between the segments of geodesic $[y, x]$ and $[y, z]$.

The law of cosines state that

$$d^2(x, z) \geq d^2(y, x) + d^2(y, z) - 2d(y, x)d(y, z) \cos \angle_y(x, z), \forall x, y, z \in M.$$

Gauss's Lemma: Let $p \in M$ and let $v, w \in B_\epsilon(0) \in T_v T_p M \approx T_p M$ and $M \ni q = \exp_p v$. Then,

$$\langle d(\exp_p)_v v, d(\exp_p)_v w \rangle_q = \langle v, w \rangle_p.$$

Let M be a simply connected manifold of non-positive curvature. Let C be a complete convex set in M . The *orthogonal projection* (or simply '*projection*') is the name given to the map $\pi: M \rightarrow C$ constructed in the following proposition: (cf. [BH99, pp 176]).

Proposition 4.1. *The projection π satisfies the following properties:*

1. *For any $x \in M$ there is a unique point $\pi(x) \in C$ such that $d(x, \pi(x)) = d(x, C) = \inf_{y \in C} d(x, y)$.*
2. *If x_0 is in the geodesic segment $[x, \pi(x)]$, then $\pi(x_0) = \pi(x)$.*
3. *Given $x \notin C$, $y \in C$ and $y \neq \pi(x)$, then $\angle_{\pi(x)}(x, y) \geq \frac{\pi}{2}$.*

4.2 Geometric Marstrand

In what follows M denotes a simply connected surface with a Riemannian metric of non-positive curvature. Using Theorems 1.1 and 1.2 we show the following theorem:

Theorem 4.2 (Geometric Marstrand). *Let M be a Hadamard surface, $X \subset M$ analytic and $p \in M$ be given. Then*

1. *If $HD(X) > 1$ then, for almost every line l coming from p , we have that $\pi_l(X)$ has positive Lebesgue measure, where π_l is the orthogonal projection on l .*
2. *If $HD(X) \leq 1$ then, for almost every line l coming from p , we have that $HD(\pi_l(X)) = HD(X)$, where π_l is the orthogonal projection on l .*

Consider \mathbb{R}^2 with a Riemannian metric $\langle \cdot, \cdot \rangle$, such that the curvature $K_{\mathbb{R}^2}$ is non-positive, i.e., $K_{\mathbb{R}^2} \leq 0$. Recall that a line γ in \mathbb{R}^2 is a geodesic defined for all parameter values and minimizing distance between any of its points, that is, $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$ and $d(\gamma(t), \gamma(s)) = |t - s|$, where d is the distance induced by the Riemannian metric $\langle \cdot, \cdot \rangle$. In other words,

there is a parametrization of γ which is an isometry. Then, given $x \in \mathbb{R}^2$ there is a unique $\gamma(t_x)$ such that $\pi_\gamma(x) = \gamma(t_x)$. Slightly abusing of the notation, we will write $\pi_\gamma(x) = t_x$.

Fix $p \in \mathbb{R}^2$ and let $\{e_1, e_2\}$ be a positive orthogonal basis of $T_p\mathbb{R}^2$, i.e., the basis $\{e_1, e_2\}$ has the induced orientation of \mathbb{R}^2 . Then, call $v_\lambda = (\cos \lambda, \sin \lambda)$ in coordinates the unit vector $(\cos \lambda)e_1 + (\sin \lambda)e_2 \in T_p\mathbb{R}^2$. Denote by l_λ the line through p with velocity v_λ , given by $l_\lambda(s) = \exp_p s v_\lambda$ and by π_λ the projection on l_λ . Then, given $\lambda \in [0, 2\pi)$, we can define $\pi: [0, 2\pi) \times T_p\mathbb{R}^2 \rightarrow \mathbb{R}$ by the unique parameter s such that $\pi_\lambda(\exp_p w) = \exp_p s v_\lambda$ i.e., $\pi(\lambda, w) := \pi_\lambda(w)$ and

$$\pi_\lambda(\exp_p w) = \exp_p \pi(\lambda, w) v_\lambda.$$

Then using the Hadamard's theorem (cf. [dC08]), the previous theorem can be stated as follows:

Theorem 4.3. *Let \mathbb{R}^2 be endowed with a metric $\langle \cdot, \cdot \rangle$ of non-positive curvature, and $K \subset \mathbb{R}^2$ analytic. Then:*

1. *If $HD(K) > 1$ then, for almost every $\lambda \in [0, \pi)$, we have that $m(\pi_\lambda(K)) > 0$, where m is the Lebesgue measure.*
2. *If $HD(K) \leq 1$ then, for almost every $\lambda \in [0, \pi)$, we have that $HD(\pi_\lambda(K)) = HD(K)$, where π_l is the orthogonal projection on l .*

The proof of this theorem will be a consequence of Lemma 4.5 below and Theorems 1.1 and 1.2.

Remark 4.4. Consider the closed ball $B_R(p) \subset \mathbb{R}^2$ of radius R and center p , then there is a constant $c \geq 0$ such that

$$-c \leq \kappa(x) \leq 0 \quad \text{for } x \in B_R(p), \quad (4.1)$$

where $\kappa(x)$ is the Gaussian curvature of x . Therefore, the equation 4.1 implies that there is a constant $k > 0$ such that for each $x \in B_R(p)$ and $u, v \in T_x\mathbb{R}^2$ we have that (cf. [dC08])

$$k \|u - v\| \geq d(\exp_x u, \exp_x v) \geq \|u - v\|. \quad (4.2)$$

Given $u \neq v$, consider γ_{uv} the unique geodesic such that $\gamma_{uv}(0) = u$ and $\gamma_{u,v}(d(u, v)) = v$, then there are unique $\theta(u, v) \in [0, 2\pi)$, $t(u, v), s(u, v) \in \mathbb{R}$ such that the geodesic $l_{\theta(u, v)}(s) = \exp_p s v_{\theta(u, v)}$ intersect orthogonally the geodesic γ_{uv} in the point

$$I(u, v) := \exp_p s(u, v) v_{\theta(u, v)} = \gamma_{uv}(t(u, v))$$

and the basis $\{l'_{\theta(u, v)}(s(u, v)) = d(\exp_p)_{s(u, v) v_{\theta(u, v)}}(v_{\theta(u, v)}), \gamma'_{uv}(t(u, v))\}$ is positive. Note that $\theta(u, v)$ is defined even if $\gamma(u, v)$ passes through p .

For $x \neq p$ denote by $\theta(x)$ the angle such that the geodesic $\exp_p t v_{\theta(x)}$ passes through x in positive time.

Lemma 4.5. *Let $K \subset \mathbb{R}^2$ a compact set, then there is a positive constant η (which only depends on K) such that for any $u, v \in K$*

$$d(\pi_{\lambda+\theta(u, v)}(u), \pi_{\lambda+\theta(u, v)}(v)) \geq \eta |\sin(\lambda)| d(u, v) \quad \text{for any } \lambda \in [0, \pi].$$

We will use this Lemma to prove Theorem 4.3.

Remark 4.6. In the inequality of Lemma 4.5, taking $\beta = \lambda + \theta(u, v)$, we get

$$d(\pi_\beta(u), \pi_\beta(v)) \geq \eta |\sin(\beta - \theta(u, v))| d(u, v) \quad \text{for any } \beta \in [\theta(u, v), \theta(u, v) + \pi]. \quad (4.3)$$

Proof of Theorem 4.3. By Corollary 7 in [How95], we may assume without loss of generality that K is compact. With the notation of Theorem 1.2, consider the probability space $([0, \pi), m)$ where m is the normalized Lebesgue measure, $X = K$ and $Y = \mathbb{R}$. We want to prove that we have the transversality condition (1.1) in this case, taking $\kappa = \dim Y = 1$ and $\alpha = 1$.

Claim: For every $\lambda \in [0, \pi)$ and $u, v \in K$, there is $\theta_\lambda(u, v) \in [\theta(u, v), \theta(u, v) + \pi]$ such that

$$d(\pi_\lambda(u), \pi_\lambda(v)) = d(\pi_{\theta_\lambda(u, v)}(u), \pi_{\theta_\lambda(u, v)}(v)). \quad (4.4)$$

Proof of Claim: We exhibit explicitly $\theta_\lambda(u, v)$. In fact:

1.

$$0 \leq \theta(u, v) < \pi : \quad \theta_\lambda(u, v) = \begin{cases} \lambda + \pi, & \text{if } 0 \leq \lambda < \theta(u, v) \\ \lambda, & \text{if } \theta(u, v) \leq \lambda \leq \pi \end{cases}.$$

2.

$$\pi \leq \theta(u, v) < 2\pi : \quad \theta_\lambda(u, v) = \begin{cases} \lambda, & \text{if } 0 \leq \lambda < \theta(u, v) - \pi \\ \lambda + \pi, & \text{if } \theta(u, v) - \pi \leq \lambda \leq \pi \end{cases}.$$

This $\theta_\lambda(u, v)$ satisfies the condition of the claim, and the proof of the claim is finished.

The above claim implies that

$$\begin{aligned} d(\pi_\lambda(u), \pi_\lambda(v)) &= d(\pi_{\theta_\lambda(u, v)}(u), \pi_{\theta_\lambda(u, v)}(v)) \\ &\geq \eta |\sin(\pi_{\theta_\lambda(u, v)} - \theta(u, v))| d(u, v) \quad \text{by Remark 4.6} \\ &\geq \eta |\sin(\lambda - \theta(u, v))| d(u, v). \end{aligned}$$

Therefore, the last inequality implies that

$$\begin{aligned} m[\lambda \in [0, \pi) : d(\pi_\lambda(u), \pi_\lambda(v)) \leq \delta d(u, v)] &\leq m[\lambda \in [0, \pi) : \eta |\sin(\lambda - \theta(u, v))| \leq \delta] \\ &= m[\lambda \in [0, \pi) : |\sin(\lambda - \theta(u, v))| \leq \delta/\eta]. \end{aligned}$$

It is easy to see that there is a constant $C > 0$ such that for all $\delta > 0$

$$m[\lambda \in [0, \pi) : |\sin(\lambda - \theta(u, v))| \leq \delta/\eta] \leq C\delta,$$

so the transversality condition (1.1) is satisfied, therefore, since $HD(K) = \dim X > 1 = \alpha\kappa$, by Theorem 1.2 (i) we conclude the proof of the theorem. \square

4.2.1 Proof of Lemma 4.5

Before proving this lemma we will give some definitions and state some auxiliary lemmas.

Definition 4.7. Given $x \neq y$, if the geodesic γ_{xy} does not pass through p , then for $\epsilon > 0$ small we can define the function

$$\alpha_y(x, \lambda) := \angle_x(\pi_{\theta(x,y)+\lambda}(x), y) \quad \text{and} \quad \alpha_x(y, \lambda) := \angle_y(\pi_{\theta(x,y)+\lambda}(y), x)$$

for all $|\lambda| < \epsilon$.

In this case, denote by $I(x, y, \lambda)$ the unique point of intersection of the geodesic $\exp_{p^{\text{sv}}\theta(x,y)+\lambda}$ with the geodesic γ_{xy} (cf. Figures 1a, 1b). On the other hand, if the geodesic γ_{xy} passes through p (cf. figure 2), then for $\epsilon > 0$ small and $|\lambda| < \epsilon$ we define the function

$$\alpha_p(x, \lambda) := \angle_x(\pi_{\theta(x,y)+\lambda}(x), p) \quad \text{and} \quad \alpha_p(y, \lambda) := \angle_y(\pi_{\theta(x,y)+\lambda}(y), p).$$

Lemma 4.8. *The functions $(x, y) \rightarrow \pi_{\theta(x,y)+\lambda}(x), \pi_{\theta(x,y)+\lambda}(y)$ are differentiable. If γ_{xy} does not pass through p , then $I(x, y, \lambda)$ is also differentiable in λ .*

Proof. The differentiability of $\pi_{\theta(x,y)+\lambda}(x)$ and $\pi_{\theta(x,y)+\lambda}(y)$ is an immediate consequent of the convexity of the distant function and Theorem of Implicit Functions.

Let us prove the second part. Since γ_{xy} does not pass through p , then there is a neighbourhood $U(x, y) \subset [0, 2\pi)$ of $\theta(x, y)$ such that $I(x, y, \lambda)$ is a Poincaré-like map between $U(x, y)$ and $\gamma_{xy}(\mathbb{R})$ defined as the intersection of the geodesic γ_{v_λ} with γ_{xy} for $\lambda \in U(x, y)$. Therefore, there is $\epsilon > 0$ small such that $(\theta(x, y) - \epsilon, \theta(x, y) + \epsilon) \subset U(x, y)$. Thus we conclude the proof of second part. \square

Lemma 4.9. *For $x, y \in K$, $x \neq y$, we have*

1. *If the geodesic γ_{xy} does not pass through p , then the functions $\alpha_y(x, \lambda)$ and $\alpha_x(y, \lambda)$ are differentiable in λ . Moreover,*

$$\lim_{\lambda \rightarrow 0} \frac{\alpha_y(x, \lambda)}{\lambda} = 1 \quad \text{and} \quad \lim_{\lambda \rightarrow 0} \frac{\alpha_x(y, \lambda)}{\lambda} = 1.$$

2. *If the geodesic γ_{xy} passes through p , then the functions $\alpha_p(x, \lambda)$ and $\alpha_p(y, \lambda)$ are differentiable in λ . Moreover,*

$$\lim_{\lambda \rightarrow 0} \frac{\alpha_p(x, \lambda)}{\lambda} = 1 \quad \text{and} \quad \lim_{\lambda \rightarrow 0} \frac{\alpha_p(y, \lambda)}{\lambda} = 1.$$

Proof. In both case the differentiability is due to fact that exponential map is differentiable and Lemma 4.8.

1. Suppose that the geodesic γ_{xy} does not pass through p (cf. Figure 1). Consider the triangle $\Delta_1(\lambda)$ generated by the points $\{p, I, I(x, y, \lambda)\}$ and let $m(\lambda) = \angle_{I(x,y,\lambda)}(p, I)$ for $|\lambda| < \epsilon$, so, by Lemma 4.8, since $I(x, y, \lambda)$ is differentiable in λ , the function $m(\lambda)$ is differentiable in λ . Consider the function

$$h_1(\lambda) = \int_{\Delta_1(\lambda)} \kappa(x) d\sigma(x),$$

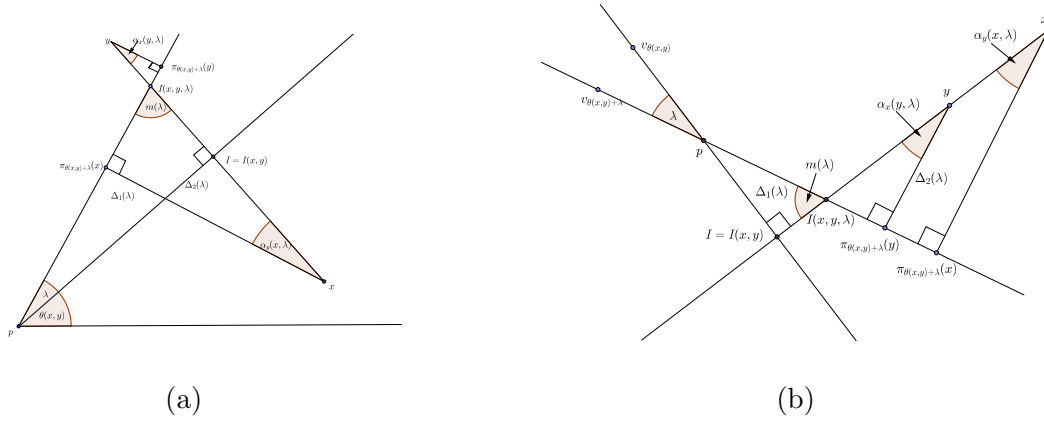


Figure 1: γ_{xy} does not pass through of p

where $\kappa(x)$ denotes the Gaussian curvature. Thus, Gauss-Bonnet theorem implies that

$$h_1(\lambda) = \int_{\Delta_1(\lambda)} \kappa(x) d\sigma(x) = -\pi/2 + \lambda + m(\lambda),$$

which is differentiable since $m(\lambda)$ is differentiable. Moreover, since $\kappa(x) \leq 0$ then $h_1(\lambda)$ has a maximum in $\lambda = 0$, thus $h'_1(0) = 0$. So, we have that $0 = h'_1(0) = m'(0) + 1$. If we consider now the triangle $\Delta_2(\lambda)$ generated by the points $\{x, I(x, y, \lambda), \pi_{\theta(x, y)+\lambda}(x)\}$, similarly, $\alpha_y(x, \lambda)$ is differentiable in λ , since Lemma 4.8 and its definition. Thus, Gauss-Bonnet theorem implies that $h_2(\lambda) = \int_{\Delta_2(\lambda)} \kappa(x) d\sigma(x)$ is differentiable. Moreover, since $\kappa(x) \leq 0$ then $h_2(\lambda)$ has a maximum in $\lambda = 0$, thus $h'_2(0) = 0$, which implies that

$$0 = h'_2(0) = m'(0) + \lim_{\lambda \rightarrow 0} \frac{\alpha_y(x, \lambda)}{\lambda}.$$

Therefore, $\lim_{\lambda \rightarrow 0} \frac{\alpha_y(x, \lambda)}{\lambda} = 1$. The proof for $\alpha_x(y, \lambda)$ is analogue.

2. Let's assume now that the geodesic γ_{xy} passes through p (cf. Figure 2).

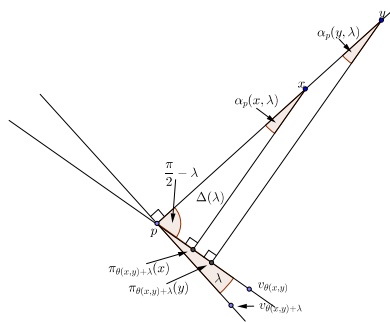


Figure 2: γ_{xy} passes through of p

In this case, consider the triangle $\Delta(\lambda)$ generated by the points $\{p, x, \pi_{\theta(x,y)+\lambda}(x)\}$. Analogously as above, the function

$$h(\lambda) = \int_{\Delta(\lambda)} \kappa(x) d\sigma(x) = -\frac{\pi}{2} + \alpha_p(x, \lambda) + \frac{\pi}{2} - \lambda$$

is differentiable, with maximum in $\lambda = 0$, then $\lim_{\lambda \rightarrow 0} \frac{\alpha_p(x, \lambda)}{\lambda} = 1$.

The proof for $\alpha_p(y, \lambda)$ is analogous.

□

Corollary 4.10. *There is $\epsilon_1 > 0$ such that for all $x, y \in K$, $x \neq y$ and all $|\lambda| \leq \epsilon_1$ we have*

1. *If the geodesic γ_{xy} does not pass through p , then*

$$\sin \alpha_y(x, \lambda) \geq \frac{1}{2} \sin |\lambda| \quad \text{and} \quad \sin \alpha_x(y, \lambda) \geq \frac{1}{2} \sin |\lambda|.$$

2. *If the geodesic γ_{xy} passes through p , then*

$$\sin \alpha_p(x, \lambda) \geq \frac{1}{2} \sin |\lambda| \quad \text{and} \quad \sin \alpha_p(y, \lambda) \geq \frac{1}{2} \sin |\lambda|.$$

Lemma 4.11. *Given three different points x, y, z then*

$$d(x, y) \geq \sin \angle_z(x, y) d(x, z).$$

Proof. By the law of cosines

$$\begin{aligned} \left(\frac{d(x, y)}{d(x, z)} \right)^2 &\geq \left(\frac{d(y, z)}{d(x, z)} \right)^2 - 2 \frac{d(y, z)}{d(x, z)} \cos \angle_z(x, y) + 1 \\ &= \left(\frac{d(y, x)}{d(x, z)} - \cos \angle_z(x, y) \right)^2 + 1 - \cos^2 \angle_z(x, y) \\ &\geq \sin^2 \angle_z(x, y). \end{aligned}$$

□

Lemma 4.12. *Given three different points $x, y, z \in B_R(p)$ with $\angle_y(x, z) = \pi/2$, let $\tilde{y}_x = \exp_x^{-1}y$, $\tilde{z}_x = \exp_x^{-1}(z)$, $\tilde{x}_z = \exp_z^{-1}x$ and $\tilde{y}_z = \exp_z^{-1}y$, then there is $\gamma > 0$ such that*

$$\sin \angle_{\tilde{y}_x}(0, \tilde{z}_x) \geq \gamma \quad \text{and} \quad \sin \angle_{\tilde{y}_z}(0, \tilde{x}_z) \geq \gamma.$$

Proof. Call L the geodesic orthogonal to γ_{xy} in y . Let $V(x, y)$ be such that $\exp_x V(x, y) = y$ and put $v(x, y) = \frac{V(x, y)}{\|V(x, y)\|}$ and $v_\lambda(x, y) \in T_x \mathbb{R}^2$ the vector that forms a angle λ with $v(x, y)$. The function $f(\lambda)$ is defined as the unique point of intersection of the geodesic $\exp_x s v_\lambda(x, y)$ with the geodesic L . Thus, there exists a unique $s(\lambda)$ such that $f(\lambda) = \exp_x s(\lambda) v_\lambda(x, y)$. Then, as in the the proof of second part of Lemma 4.8, the function f is differentiable in λ . Consider the function $g(\lambda) = s(\lambda) v_\lambda(x, y) = \exp_x^{-1}(f(\lambda))$ which is

also differentiable. If $\theta(x, y, \lambda)$ denotes the angle between $g(0) = V(x, y)$ and $g(\lambda) - g(0)$, then

$$\lim_{\lambda \rightarrow 0^\pm} \cos \theta(x, y, \lambda) = \pm \frac{\langle g(0), g'(0) \rangle_x}{\|g(0)\| \|g'(0)\|} = 0.$$

Thus, since $x, y \in B_R(p)$, the above equation implies that there are $\tilde{\epsilon} > 0$ and $\tilde{\gamma} > 0$ independent of x, y such that

$$\sin \theta(x, y, \lambda) > \tilde{\gamma} \quad \text{for } |\lambda| < \tilde{\epsilon} \quad \text{and } x, y \in B_R(p).$$

Since $\angle_y(x, z) = \frac{\pi}{2}$, then by definition of $g(\lambda)$ we have that $g(\lambda) \geq g(0)$, which implies that $\frac{\|g(\lambda)\|}{\|g(\lambda) - g(0)\|} \geq 1$. Therefore, if $|\lambda| \geq \tilde{\epsilon}$, then, by the law of sines,

$$\sin(\theta(x, y, \lambda)) = \frac{\|g(\lambda)\|}{\|g(\lambda) - g(0)\|} \sin |\lambda| \geq \sin |\lambda| \geq \sin \tilde{\epsilon}.$$

Taking $\gamma = \max\{\sin \tilde{\epsilon}, \tilde{\gamma}\}$, we conclude the proof noting that $x, y, z \in B_R(p)$ and $z \in L$. \square

Remark 4.13. Observe that the previous Lemma holds if $|\angle_y(x, z) - \pi/2| < \delta$ for some $\delta \in (0, \frac{\pi}{2})$, because we only need that $|\cos \angle_y(x, z)| < a < 1$ for some $a > 0$.

Lemma 4.14. *Let $x, y, z \in B_R(p)$ three different points. Suppose that $\angle_y(x, z) = \pi/2$. Then*

$$\frac{\gamma}{k} d(x, y) \leq d(z, x) \sin \angle_z(x, y) \leq d(x, y),$$

where γ, k are as in Lemma 4.12 and Remark 4.4, respectively.

Proof. The right side of the inequality is given by Lemma 4.11. Let us prove the left side of the inequality. Consider the triangle in $T_z \mathbb{R}^2$ determined by the points $0, \tilde{y}_z = \exp_z^{-1} y$ and $\tilde{x}_z = \exp_z^{-1} x$. Then, by law of sines, we have

$$\frac{\sin \angle_z(x, y)}{\|\tilde{x}_z - \tilde{y}_z\|} = \frac{\sin \angle_{\tilde{y}_z}(0, \tilde{x}_z)}{\|\tilde{x}_z\|} = \frac{\sin \angle_{\tilde{x}_z}(0, \tilde{y}_z)}{\|\tilde{y}_z\|}.$$

Moreover, by Lemma 4.12 there is $\gamma > 0$ with $\sin \angle_{\tilde{y}_z}(0, \tilde{x}_z) \geq \gamma > 0$, so

$$\|\tilde{x}_z\| \sin \angle_z(x, y) \geq \gamma \|\tilde{x}_z - \tilde{y}_z\|.$$

Since $\|\tilde{x}_z\| = d(z, x)$, and by equation (4.2), $\|\tilde{x}_z - \tilde{y}_z\| \geq \frac{1}{k} d(x, y)$, so

$$\frac{\gamma}{k} d(x, y) \leq d(z, x) \sin \angle_z(x, y).$$

\square

Proof of Lemma 4.5. Without loss of generality we can assume that $K \subset B_R(p)$.

Suppose that $u \neq v$ and the geodesic γ_{uv} does not pass through of point p . Without loss of generality, we can assume that $\theta(u, v) = 0$. Denote $I(\lambda) = I(u, v, \lambda)$ and $I = I(u, v, 0)$. Let us subdivide the proof in two cases:

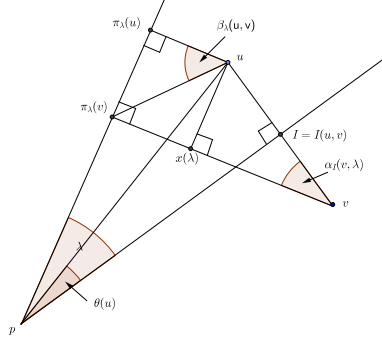


Figure 3: Case 1, $I \in [u, v]$

1. $I \in [u, v]$,
2. $I \notin [u, v]$.

Let us first prove the case 1 (cf. Figure 3). In fact, by Lemma 4.11,

$$d(\pi_\lambda(v), I(\lambda)) \geq \sin \alpha_I(v, \lambda) d(I(\lambda), v),$$

$$d(\pi_\lambda(u), I(\lambda)) \geq \sin \alpha_I(u, \lambda) d(I(\lambda), u).$$

Let ϵ_1 be given by Corollary 4.10. Then, for $|\lambda| < \epsilon_1$, we have

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \frac{1}{2} \sin |\lambda| d(u, v). \quad (4.5)$$

Denote by C_{ϵ_1} the cone with angle ϵ_1 , centered on the geodesic with initial velocity $v_{\theta(u,v)}$ (that in this case is equal to e_1 , the first vector of the canonical basis in $T_p \mathbb{R}^2$). Suppose that $[u, v] \subset C_{\epsilon_1}$, then the equation (4.5) is satisfied for any λ such that the geodesic with initial velocity v_λ intersects $[u, v]$. If $|\lambda| \leq \epsilon_1$ and the geodesic with initial velocity v_λ does not intersect $[u, v]$, then we consider the quadrilateral generated by the point $u, v, \pi_\lambda(u)$ and $\pi_\lambda(v)$. Without loss of generality, we can assume that $d(\pi_\lambda(u), u) \leq d(\pi_\lambda(v), v)$ and consider $x(\lambda) = \pi_{[\pi_\lambda(v), v]}(u)$, the projection of u over the of geodesic $[\pi_\lambda(v), v]$ (cf. Figure 3). Then, by Lemma 4.11, if $\beta_\lambda(u, v) = \angle_u(\pi_\lambda(u), \pi_\lambda(v))$, then

$$\begin{aligned} d(\pi_\lambda(u), \pi_\lambda(v)) &\geq \sin \beta_\lambda(u, v) d(\pi_\lambda(v), u) \\ &\geq \sin \beta_\lambda(u, v) d(x(\lambda), u) \\ &\geq \sin \beta_\lambda(u, v) \sin \alpha_u(v, \lambda) d(u, v) \\ &\geq \frac{1}{2} \cdot \sin \beta_\lambda(u, v) \sin |\lambda| d(u, v). \end{aligned} \quad (4.6)$$

Thus, by Lemma 4.14, we have

$$\frac{\gamma}{k} \cdot \frac{d(\pi_\lambda(u), \pi_\lambda(v))}{d(\pi_\lambda(v), u)} \leq \sin \beta_\lambda(u, v) \leq \frac{d(\pi_\lambda(u), \pi_\lambda(v))}{d(\pi_\lambda(v), u)}.$$

Since, $d(\pi_\lambda(u), \pi_\lambda(v)) \rightarrow d(\pi_{\theta(u)}(v), u)$ and $d(\pi_\lambda(v), u) \rightarrow d(\pi_{\theta(u)}(v), u)$ as $\lambda \rightarrow \theta(u)$, then there is $\delta_1 > 0$ such that

$$\sin \beta_\lambda(u, v) \geq \frac{\gamma}{2k} \quad \text{for } 0 \leq \lambda - \theta(u) \leq \delta_1.$$

Therefore, by equation (4.6),

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \frac{\gamma}{4k} \sin |\lambda| d(u, v) \quad \text{for } 0 \leq \lambda - \theta(u) \leq \delta_1. \quad (4.7)$$

Suppose now that $[u, v] \not\subset C_{\epsilon_1}$. Then, diminishing ϵ_1 if necessary, we can assume that $C_{\epsilon_1} \cap [u, v]$ is contained in the interior $\text{int } [u, v]$ of $[u, v]$. Thus if $|\lambda| \geq \epsilon_1$, there are $\alpha > 0$ and $\eta_1 > 0$ such that $\alpha_I(i, \lambda) \geq \alpha > 0$ for $i = u, v$, and

$$\sin \alpha_I(i, \lambda) \geq \eta_1 \sin \lambda \quad i = u, v \quad \text{and} \quad v_\lambda \in C(u, v),$$

where $C(u, v) = \{sv_\theta : s \in \mathbb{R}, \theta \in [\theta(v), \theta(u)]\}$ is the cone generated by u and v . Therefore,

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \eta_1 \sin(\lambda) d(u, v) \quad \text{for } |\lambda| \geq \epsilon_1 \quad \text{and} \quad v_\lambda \in C(u, v). \quad (4.8)$$

If $|\lambda| \geq \epsilon_1$ and $v_\lambda \notin C(u, v)$, then, as above, since $d(\pi_\lambda(u), \pi_\lambda(v)) \rightarrow d(\pi_{\theta(u)}(v), u)$ and $d(\pi_\lambda(v), u) \rightarrow d(\pi_{\theta(u)}(v), u)$ as $\lambda \rightarrow \theta(u)$, by Lemma 4.14 we have that there are δ_2 and η_2 such that

$$\sin \beta_\lambda(u, v) \geq \frac{\gamma}{2k} \quad \text{and} \quad \sin \alpha_u(v, \lambda) \geq \eta_2 \sin \lambda \quad \text{for } |\lambda - \theta(u)| \leq \delta_2.$$

Thus, as in the equation (4.6)

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \frac{\gamma \eta_2}{2k} \sin(\lambda) d(u, v) \quad \text{for } 0 \leq \lambda - \theta(u) \leq \delta_2. \quad (4.9)$$

Therefore, in any case $[u, v] \subset C_{\epsilon_1}$ or $C_{\epsilon_1} \cap [u, v] \subset \text{int } [u, v]$. Taking $\delta_3 = \min\{\delta_1, \delta_2\}$, $\Gamma_1 = \max\{\frac{\gamma}{4k}, \frac{\gamma \eta_2}{2k}\}$ and $\Gamma_2 = \max\{\frac{\gamma}{4k}, \eta_1\}$ the equations (4.7), (4.8) and (4.9) give us that

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \Gamma_1 \sin \lambda d(u, v) \quad \text{for } 0 \leq \lambda - \theta(u) \leq \delta_3. \quad (4.10)$$

and

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \Gamma_2 \sin \lambda d(u, v) \quad \text{for } v_\lambda \in C(u, v). \quad (4.11)$$

It is worth noting that equation (4.10) holds for $\lambda > 0$ and $0 \leq \theta(-v) - \lambda \leq \delta_3$.

Thus, taking $\Gamma_3 = \max\{\Gamma_1, \Gamma_2\}$, for

$\lambda \in C(u, v, \delta_3) := (0, \theta(u) + \delta_3) \cup (\theta(-v) - \delta_3, \pi)$, we have

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \Gamma_3 \sin(\lambda) d(u, v). \quad (4.12)$$

Now let us consider $\lambda \notin C(u, v, \delta_3)$. We consider the quadrilateral generated by the point $u, v, \pi_\lambda(u)$ and $\pi_\lambda(v)$. Without loss of generality, we can assume that $d(\pi_\lambda(u), u) \leq d(\pi_\lambda(v), v)$ and consider $x(\lambda) = \pi_{[\pi_\lambda(v), v]}(u)$, the projection of u over the geodesic $[\pi_\lambda(v), v]$. Then, by Lemma 4.11 (cf. Figure 3),

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \sin \beta_\lambda(u, v) \sin \alpha_u(v, \lambda) d(u, v).$$

By Lemma 4.14 there is $\eta_3 > 0$ such that $\sin \beta_\lambda(u, v) \geq \eta_3$ and, since δ_3 is a constant that does not depend on u and v , there is $\eta_4 > 0$ such that $\sin \alpha_u(v, \lambda) \geq \eta_4 \sin \lambda$ for $\lambda \notin C(u, v, \delta_3)$, so

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \eta_3 \cdot \eta_4 \sin \lambda d(u, v). \quad (4.13)$$

This last equation together with equation (4.12) prove the lemma in the case 1 ($I \in [u, v]$).

For the other case ($I \notin [u, v]$, cf. figure 4), without loss of generality we can assume that $\theta(u, v) = 0$, and for λ small it holds that $d(u, \pi_\lambda(u)) \geq d(v, \pi_\lambda(v))$. Denote $x(\lambda) = \pi_{[u, \pi_\lambda(u)]}(v)$, the projection of v over the segment of geodesic $[u, \pi_\lambda(u)]$. Then, by Lemma 4.11,

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \sin \beta(\lambda) d(\pi_\lambda(v), x(\lambda)),$$

where $\beta(\lambda) = \angle_{x(\lambda)}(\pi_\lambda(u), \pi_\lambda(v))$.

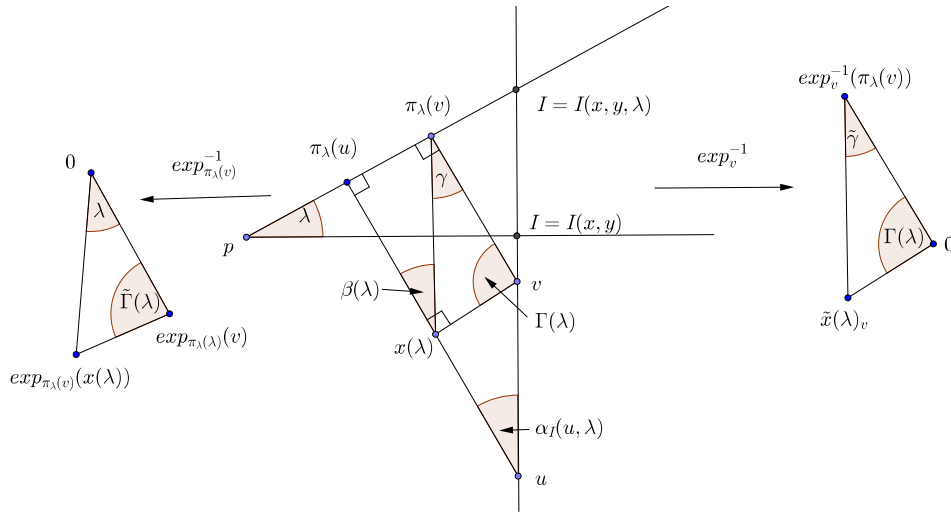


Figure 4: case 2 ($I \notin [u, v]$)

Considering the triangle in $T_v\mathbb{R}^2$ given by the points 0 , $\tilde{x}(\lambda)_v = \exp_v^{-1}(x(\lambda))$ and $\exp_v^{-1}(\pi_\lambda(v))$, then, by the law of sines, $\sin \tilde{\gamma}(\lambda) = \frac{d(x(\lambda), v) \sin \Gamma(\lambda)}{\|\tilde{x}(\lambda)_v - \exp_v^{-1}(\pi_\lambda(v))\|}$, where $\tilde{\gamma}(\lambda) = \angle_{\exp_v^{-1}(\pi_\lambda(v))}(0, \tilde{x}(\lambda)_v)$ and $\Gamma(\lambda) = \angle_v(\pi_\lambda(v), x(\lambda))$.

Since $\|\tilde{x}(\lambda)_v - \exp_v^{-1}(\pi_\lambda(v))\| \leq d(\pi_\lambda(v), x(\lambda))$ we have that

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \frac{\sin \beta(\lambda)}{\sin \tilde{\gamma}(\lambda)} \sin \Gamma(\lambda) d(x(\lambda), v).$$

Using Lemma 4.11 and Corollary 4.10 we have that

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \frac{1}{2} \frac{\sin \beta(\lambda)}{\sin \tilde{\gamma}(\lambda)} \sin \Gamma(\lambda) \sin |\lambda| d(u, v) \quad \text{for } |\lambda| \leq \epsilon_1.$$

Now, since $\angle_{x(\lambda)}(v, \pi_\lambda(u)) = \angle_{\pi_\lambda(u)}(x(\lambda), \pi_\lambda(v)) = \angle_{\pi_\lambda(v)}(v, \pi_\lambda(u)) = \pi/2$, Remark 4.4 and Gauss Bonnet Theorem imply that $\Gamma(\lambda) \rightarrow \pi/2$ as $\lambda \rightarrow 0$. Thus, we can assume that

there is a constant c_1 such that

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \frac{c_1 \sin \beta(\lambda)}{2 \sin \tilde{\gamma}(\lambda)} \sin |\lambda| d(u, v) \quad \text{for } |\lambda| \leq \epsilon_1. \quad (4.14)$$

Note that $\gamma(\lambda) - (\pi/2 - \Gamma(\lambda)) \leq \beta(\lambda)$, where $\gamma(\lambda) = \angle_{\pi_\lambda(v)}(v, x(\lambda))$, thus, since the sine function is increasing and by the above property of $\Gamma(\lambda)$ we have that $\sin \beta(\lambda) \geq \frac{\sin \gamma(\lambda)}{2}$ for $|\lambda| \leq \epsilon_1$.

Considering the triangle in $T_{\pi_\lambda(v)}\mathbb{R}^2$ given by the points 0 , $\exp_{\pi_\lambda(v)}^{-1}(x(\lambda))$ and $\exp_{\pi_\lambda(v)}^{-1}(v)$, if $\tilde{\Gamma}(\lambda) = \angle_{\exp_{\pi_\lambda(v)}^{-1}(v)}(0, \exp_{\pi_\lambda(v)}^{-1}(x(\lambda)))$ then, by the law of sines and equation (4.2) we have

$$\frac{\sin \gamma(\lambda)}{\sin \tilde{\gamma}(\lambda)} \geq \frac{1}{k^2} \cdot \frac{\sin \tilde{\Gamma}(\lambda)}{\sin \Gamma(\lambda)} \geq \frac{c_2}{k^2} \quad \text{for } |\lambda| \leq \epsilon_1,$$

for some positive constant c_2 (since $\Gamma(\lambda) \rightarrow \frac{\pi}{2}$ as $\lambda \rightarrow 0$ and by Remark 4.13 of Lemma 4.12). By the above inequality and equation (4.14), we have

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \frac{c_1 c_2}{4k^2} \sin(\lambda) d(u, v) \quad \text{for } 0 \leq \lambda \leq \epsilon_1 \quad \text{and} \quad \pi - \epsilon_1 \leq \lambda \leq \pi. \quad (4.15)$$

For $\lambda \in (\epsilon_1, \pi - \epsilon_1)$, we have (with the same notation above)

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq d(\pi_\lambda(v), x(\lambda)) \sin \beta(\lambda) \geq \sin \beta(\lambda) \sin \alpha_u(v, \lambda) d(u, v).$$

Since $\lambda \in (\epsilon_1, \pi - \epsilon_1)$ (similarly as we got in equation (4.13)), there are two constants $c_3 > 0$ and $c_4 > 0$ such that $\sin \beta(\lambda) \geq c_3$ and $\sin \alpha_u(v, \lambda) \geq c_4 \sin \lambda$, thus

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq c_3 c_4 \sin(\lambda) d(u, v).$$

Taking $\eta = \max\{\frac{c_1 c_2}{4k^2}, c_3 c_4\}$, we conclude that

$$d(\pi_\lambda(u), \pi_\lambda(v)) \geq \eta \sin(\lambda) d(u, v).$$

If the geodesic γ_{uv} passes through the point p , the proof is analogous to the second case above. \square

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